

Aggregation of autoregressive processes and long memory

By Didier Dacunha-Castelle¹ and Lisandro J. Fermín^{1,2}

¹ UMR CNRS 8628. Équipe de Probabilités, Statistique et Modélisation, Bât. 425,
Université Paris-Sud, 91405 Orsay Cedex, France.

² UMR CNRS 8145 Laboratoire de Mathématiques Appliquées à Paris 5 (MAP5),
Université Paris Descartes, 45 rue des Saints Pères 75270 Paris Cedex 06, France.

e-mail: didier.dacunha-castelle@math.u-psud.fr, fermin.lisandro@math.u-psud.fr

Abstract

We study the aggregation of AR processes and generalized Ornstein-Uhlenbeck (OU) processes. Mixture of spectral densities with random poles are the main tool. In this context, we apply our results for the aggregation of doubly stochastic interactive processes, see [4]. Thus, we study the relationship between aggregation of autoregressive processes and long memory considering complex interaction structures. We precise a very interesting qualitative phenomena: how the long memory creation depends on the poles concentration near to the boundary of stability (measured in the Prokhorov sense). Our results extends the results given by Oppenheim and Viano, [12], and highlight the importance of the angular dispersion measure of poles in the appearance of the long memory.

Keywords: Aggregation; long memory; mixture of spectral densities; *AR* processes; Ornstein-Uhlenbeck processes.

1 Introduction

Long memory (LM) processes, are used in many fields such as economics, finance, hydrology or communication networks. Some of these LM processes can be seen as an aggregation of elementary short memory (SM) processes.

The aggregation of stochastic processes was introduced by Granger in 1980, [7]. It is a summation procedure of identically distributed elementary processes $Z^i = \{Z_t^i : t \in \mathbb{Z}\}$ over the index i . Granger shows that by aggregating random parameter *AR*(1) processes, one can obtain LM processes with spectral density equivalent to $|\lambda|^{-d}$ when $\lambda \rightarrow 0$ for some d , $0 < d < 1$. He considers *AR*(1) processes with independent random Beta distributed parameters and gives conditions on the Beta distribution in order to obtain long memory. That has opened a new way of obtaining long range dependence time series.

As indicated by Beran, [2], this is an interesting idea from two different points of view: it allows a physical explanation of the long dependence phenomenon in several fields and it gives an easy and fast simulation method.

We develop the procedure of aggregation considering doubly stochastic *AR*(p) (or generalized Ornstein-Uhlenbeck process of order p) elementary processes $Z^i = \{Z_t(y^i, \varepsilon^i) : t \in T\}$, where $Y = \{y^i : i \in \mathbb{N}\}$ is a sequence of random variables with distribution μ on \mathbb{R}^p and $\mathcal{E} = \{\varepsilon^i : i \in \mathbb{N}\}$ is the sequence of innovations $\varepsilon^i = \{\varepsilon_t^i : t \in T\}$. Finally, $T = \mathbb{Z}$ in the case of discrete time

processes and $T \subseteq \mathbb{R}$ in the continuous time case. We assume that Y and \mathcal{E} satisfy the following assumption.

Assumption A1:

1. \mathcal{E} is an array of strong white noises, i.e. for each i , $\varepsilon^i = \{\varepsilon_t^i\}$ is an i.i.d sequence such that $\mathbb{E}[|\varepsilon_t^i|^2] = 1$.
2. Y is an i.i.d. sequence with distribution $\nu = \mu^{\otimes \mathbb{N}}$.
3. Y is independent of \mathcal{E} .

Then, we define the sequence of partial aggregations $X^N(Y) = \{X_t^N(Y) : t \in T\}$ of elementary processes $\{Z^i\}$, by

$$X_t^N(Y) = \frac{1}{B_N} \sum_{i=1}^N Z_t(y^i, \varepsilon^i), \quad (1)$$

where $\{B_N\}$ is a normalization sequence.

In general, we consider three types of innovations:

1. **Common innovation:** $\varepsilon^i = \varepsilon$ for all $i \in \mathbb{N}$.
2. **Independent innovations:** ε^i independent of ε^j for $i \neq j$.
3. **Stationary interactive innovations:** $\mathbb{E}[\varepsilon_t^i \varepsilon_s^j] = \chi(i - j)$ and $\mathbb{E}[\varepsilon_t^i \varepsilon_s^j] = 0$ for $t \neq s$, where χ is an interaction correlation.

The aggregation procedure of doubly stochastic linear processes can be also develop using mixtures of spectral densities and mixture of transfer functions as main tools, see [4]. Let $g(\lambda, y)$, $y \in \mathbb{R}^p$, be a family of spectral densities and μ be a probability on \mathbb{R}^p . We will denote by $h(\cdot, y)$ a particular square root of $g(\cdot, y)$, some time for simplicity we consider the real one. In other case, there exists a root linked with a regular representation. This is always possible for autoregressive process of order p .

We consider, for y^i fixed, that $g(\lambda, y^i)$ is the spectral density of the elementary process $Z^i = Z(y^i, \varepsilon^i)$. The mixture of the spectral densities $g(\lambda, y)$, is defined by

$$F(\lambda) = \int_{\mathbb{R}^p} g(\lambda, y) \mu(dy). \quad (2)$$

$F(\lambda)$ is a well defined spectral density if and only if

$$\int F(\lambda) d\lambda < \infty. \quad (3)$$

The mixture of transfer function given by

$$H(\lambda) = \int_{\mathbb{R}^s} h(\lambda, y) \mu(dy), \quad (4)$$

is well defined and will be called a transfer function if $|H|^2$ is a spectral density. This is a consequence of condition (3) and Jensen's inequality.

Without loss of generality for the purpose of this chapter, we assume that $\mathcal{E} = \{\varepsilon^i\}$ is a sequence of Gaussian white noise, except for some remarks.

Under some general conditions for the interaction χ we show in [4] the $\nu - a.s.$ weak convergence of $X^N(Y)$ to a process X called the aggregation process.

In the independent innovations case, we take $B_N = \sqrt{N}$, then one can show that the aggregation process exists $\nu - a.s.$ if and only if condition (3) hold. In this case the aggregation process X is Gaussian with spectral density F . Nevertheless, for the common innovation case,

taking $B_N = N$ we can prove that condition (3) implies the existence $\nu - a.s.$ of X , in this case X is a Gaussian process with spectral density $|H|^2$.

For the case of interactive innovations we have show, in [4], that in general condition (3) is a necessary and sufficient condition for the existence $\nu - a.s.$ of aggregation process X . Moreover, the limit is always a convex combination of the two extreme cases: independent innovations ($F(\lambda)$, $B_N = \sqrt{N}$) and common innovations ($H^2(\lambda)$, $B_N = N$). In general, the limit is reached for a normalization B_N which depends on the behavior of interaction χ . Fermín, [5], generalize this results for the case of non-gaussian innovations, considering a sequence $\mathcal{E} = \{\varepsilon^i\}$ of weakly dependent innovations.

The long memory of a stationary process is defined by the nonsummability of the corresponding covariance sequence $\gamma = \{\gamma(k) : k \in \mathbb{Z}\}$; i.e. when the norm $\|\gamma\|_1 = \sum_k |\gamma(k)| < \infty$ we say that the process is SM and if $\|\gamma\|_1 = \infty$ then we say that the process is LM. The long memory sequences are generally associated to the singularities of the spectral density.

The non-uniformity of $\|\gamma(\cdot, y)\|_1$, for different values of y , can generate long memory by aggregation. In the spectral density context, this phenomenon is related with the concentration of the mixture measure μ near the boundary of the existence domain for elementary processes, defined by the set of parameters y such that $g(\lambda, y)$ is a well defined spectral density; i.e. the set $\{y : \int g(\lambda, y) d\lambda < \infty\}$. For instance, in the case of $AR(1)$ processes aggregations, we have an only way of creating long memory; that is, when the measure μ of the autoregressive parameter is concentrated near $\{-1, 1\}$, which is the boundary of the domain of parameter values leading to stationary $AR(1)$.

Several authors, [3,6,9,10,13], study the long memory on the aggregation of $AR(1)$ and $AR(2)$ processes, but considering only the case in which the spectral density has an only singularity at zero, which induces long memory but without seasonal effects, for the continuous time case see [1,8,11]. In [12], the authors show how to obtain seasonally LM models from the aggregation of autoregressive processes in the discrete time case as well as in the continuous time case.

Our purpose is to clarify in a more general context, the conditions linked to the existence and the long memory property of the aggregation of AR and OU processes. The results given in this paper are an extension of the results shown in [12]. Our generalization consists in considering poles with multiplicity and with diffuse angular distribution.

We shall exhibit a new qualitative behavior: for $p \geq 2$ the long memory is not guaranteed even if the radial distribution of poles diverges in 1; the angular distribution of poles is also determinant and can counterbalance the radial distribution. For instance, for $AR(2)$ elementary processes, if $(\rho e^{i\theta})^{-1}$ is a random pole and if θ has a diffuse measure then ρ has to be very concentrated near $\{-1, 1\}$ whereas if θ has a concentrated measure near to a given frequency then conditions on ρ can be relaxed.

Results are presented as follows. In Section 2, we show that for suitable distributions of the $AR(p)$ parameters the aggregation process exists and we give the expression of its spectral density. Furthermore, we give necessary and sufficient conditions for this spectral density to have predetermined singularities which imply the long memory, we study in detail the cases of $AR(1)$ and $AR(2)$ processes and we present a general result for the case $AR(p)$. We illustrate how the long memory can "disappear" when the angular distribution of poles is diffuse. In Section 3 we extend the results given in Section 2 to the case of the aggregation of $OU(p)$ processes.

2 Aggregation of $AR(p)$ processes and long memory

Let D be the open unit disc. For $y = (y_1, \dots, y_p) \in D^p$, let $Z_t(y, \varepsilon)$ denote the autoregressive process with innovation ε satisfying

$$Z_t = - \sum_{k=1}^p a_k(y) Z_{t-k} + \sigma \varepsilon_t. \quad (5)$$

We assume that the corresponding characteristic polynomial is factorized of the following way

$$A(s, y) = 1 + \sum_{k=1}^p a_k(y) s^k = \prod_{k=1}^p (1 - y_k s).$$

So $\{1/y_k : k = 1, \dots, p\}$ is the set of random poles of the $AR(p)$ process. The process $Z_t(y, \varepsilon)$ has the $MA(\infty)$ expansion

$$Z_t(y) = \sigma \sum_{k=0}^{\infty} c_k(y) \varepsilon_{t-k}, \quad (6)$$

where $A^{-1}(s, y) = \sum_{k=0}^{\infty} c_k(y) s^k$.

Suppose that y is a random vector independent of the innovation ε and whose distribution μ has support D^p . From the independence assumption and because $\mu(|y_k| < 1) = 1$ for every k , the series in (6) converge almost surely for μ -almost all y . The induced process $Z_t(y, \varepsilon)$ is, for almost all y , a stationary $AR(p)$ process with spectral density $g(\lambda, y) = \sigma^2 |A(e^{i\lambda}, y)|^{-2}$. We take the transfer function $h(\lambda, y) = \sigma A(e^{i\lambda}, y)^{-1} = \sigma \sum_{k=0}^{\infty} c_k(y) e^{i\lambda k}$.

Since $F(\lambda) = \mathbb{E}[g(\lambda, y)]$, $H(\lambda) = \mathbb{E}[h(\lambda, y)]$ and

$$\mathbb{E}^y[|Z_t(y)|^2] = \int_{-\pi}^{\pi} g(\lambda, y) d\lambda,$$

where $\mathbb{E}^y[\cdot]$ denotes the conditional expectation given y , then we have that $Z_t(y, \varepsilon)$ exists in L^2 if and only if condition (3) holds.

We take $\{Z^i\}$ as a sequence of random parameters $AR(p)$ processes, with $Y = \{y^i\}$ a sequence of random vectors in D^p . Without loss of generality for the purpose of this paper, we assume that $\mathcal{E} = \{\varepsilon^i\}$ is a sequence of Gaussian white noise. We suppose that Y and \mathcal{E} satisfy the Assumption A1 and so we have the convergence results of Theorem 1 given in [4].

We denote by $\mathbb{AR}(p)$ the class of $AR(p)$ processes and by $\mathbb{M}(\mathbb{AR}(p))$ the class of processes that can be obtained by aggregation of elementary processes in $\mathbb{AR}(p)$.

In this section, our aim is to give a way of taking the measure μ that allows us to obtain the long memory property for X .

From now, we will consider that $A(s, y)$ is a polynomial of degree p with n real roots and $2(q - n)$ complex pairwise conjugate roots; i.e.,

$$A(s, y) = \prod_{k=1}^n (1 \pm \rho_k s)^{m_k} \prod_{k=n+1}^q [(1 - \rho_k e^{i\theta_k} s)(1 - \rho_k e^{-i\theta_k} s)]^{m_k},$$

where $n \leq q$, m_k is the multiplicity of the root y_k^{-1} , where $y_k = \rho_k e^{i\theta_k}$ with $\theta_k \in [-\pi, \pi)$, $\rho_k \in (0, 1)$ and $p = \sum_{k=1}^n m_k + 2 \sum_{j=n+1}^q m_j$. Furthermore, we will consider the following assumption.

Assumption B1: Let ρ_1, \dots, ρ_q be independent random variables, such that ρ_k has for distribution

$$dR_k(\rho) = |1 - \rho|^{d_k} \varphi_k(\rho) d\rho, \quad (7)$$

where φ_k is a bounded positive function with support $[0, 1]$, continuous in $\rho = 1$ with $\varphi_k(1) > 0$. Let $\theta_1, \dots, \theta_q$ be independent random variables and independent of ρ_1, \dots, ρ_q , such that θ_k has for distribution

$$dQ_k(\theta) = \frac{\psi_k(\theta) d\theta}{|\theta - \theta_k^0|^{\beta_k}}, \quad (8)$$

where $\beta_k \leq 1$, ψ_k is a bounded positive function with support $(-\pi, \pi]$, continuous in $\theta = \theta_k^0$ and $\psi_k(\theta_k^0) > 0$.

Remark 1 Note that when $\beta_k \rightarrow 1$ and $\psi_k(\theta) \rightarrow 0$ for $\theta \neq \theta_k^0$, then the measure Q_k converges to Dirac's delta; i.e., by convention when $\beta_k = 1$ we will consider $Q_k(\theta) = \delta(\theta - \theta_k^0)$. Thus, we consider $\beta_k = 1$ and $\theta_k^0 \in \{0, \pi\}$, for $1 \leq k \leq n$, without more details.

Then, if condition (3) is fulfilled we have that

$$F(\lambda) = \prod_{k=1}^n \int_0^1 \frac{dR_k(\rho)}{|1 - \rho e^{i(\lambda - \theta_k^0)}|^{2m_k}} \prod_{k=n+1}^q \int_0^1 \int_{-\pi}^{\pi} \frac{dR_k(\rho) dQ_k(\theta)}{[|1 - \rho e^{i(\lambda + \theta)}| |1 - \rho e^{i(\lambda - \theta)}|]^{2m_k}}. \quad (9)$$

$$H(\lambda) = \prod_{k=1}^n \int_0^1 \frac{dR_k(\rho)}{(1 - \rho e^{i(\lambda - \theta_k^0)})^{m_k}} \prod_{k=n+1}^q \int_0^1 \int_{-\pi}^{\pi} \frac{dR_k(\rho) dQ_k(\theta)}{[(1 - \rho e^{i(\lambda + \theta)})(1 - \rho e^{i(\lambda - \theta)})]^{m_k}}. \quad (10)$$

Remark 2 We study in detail the cases of independent innovations and common innovation. Since, in the case of interactive innovations the spectral density of the aggregated process is always a positive convex combination of the form $aF + b|H|^2$ then the results in this last case can be deduced from the two previous cases.

In the following we will consider that $A_\lambda \sim B_\lambda$ near $\lambda = \lambda_0$, if $\lim_{\lambda \rightarrow \lambda_0} A_\lambda/B_\lambda$ is a non-null constant. Under Assumption B1 we have the following lemmas.

Lemma 1

1. If $-1 < d_k < n_k - 1$, then near $\lambda = 0$

$$\int_0^1 \frac{|1 - \rho|^{d_k} \varphi_k(\rho) d\rho}{|1 - \rho e^{i\lambda}|^{n_k}} \sim \frac{\varphi_k(1)}{|\lambda|^{n_k-1-d_k}} \int_0^\infty \frac{u^{d_k} du}{(1+u^2)^{n_k/2}}.$$

2. If $-1 < d_k < n_k - 1$, then near $\lambda = \pi$

$$\int_0^1 \frac{|1 - \rho|^{d_k} \varphi_k(\rho) d\rho}{|1 + \rho e^{i\lambda}|^{n_k}} \sim \frac{\varphi_k(1)}{|\lambda - \pi|^{n_k-1-d_k}} \int_0^\infty \frac{u^{d_k} du}{(1+u^2)^{n_k/2}}.$$

3. If $-1 < d_k < n_k - 1$ and $\theta_k^0 \notin \{0, \pi\}$, then near $\lambda = \pm \theta_k^0$

$$\int_0^1 \frac{|1 - \rho|^{d_k} \varphi_k(\rho) d\rho}{[|1 - \rho e^{i(\lambda + \theta_k^0)}| |1 - \rho e^{i(\lambda - \theta_k^0)}|]^{n_k}} \sim \frac{\varphi_k(1) [2 \sin(\theta_k^0)]^{-n_k}}{|\lambda \mp \theta_k^0|^{n_k-1-d_k}} \int_0^\infty \frac{u^{d_k} du}{(1+u^2)^{n_k/2}}.$$

Lemma 2 Let $\alpha_k < 1$ and

$$f_k(\lambda) = \int_{-\pi}^{\pi} \int_0^1 \frac{|1 - \rho|^{d_k} \varphi_k(\rho) \psi_k(\theta) |\theta - \theta_k^0|^{-\alpha_k} d\rho d\theta}{[|1 - \rho e^{i(\lambda + \theta)}| |1 - \rho e^{i(\lambda - \theta)}|]^{n_k}}.$$

1. If $n_k - 1 < d_k < 2n_k - 2 + \alpha_k$, $\theta_k^0 = 0$, then near $\lambda = 0$

$$f_k(\lambda) \sim \frac{\varphi_k(1) \psi_k(0)}{|\lambda|^{2n_k-2-d_k+\alpha_k}} \int_{-\infty}^{\infty} \int_0^\infty \frac{u^{d_k} |\theta|^{-\alpha_k} du d\theta}{[(\theta - 1)^2 + u^2][(\theta + 1)^2 + u^2]^{n_k/2}}.$$

2. If $n_k - 1 < d_k < 2n_k - 2 + \alpha_k$, $\theta_k^0 = \pi$, then near $\lambda = \pi$

$$f_k(\lambda) \sim \frac{\varphi_k(1) \psi_k(\pi)}{|\lambda - \pi|^{2n_k-2-d_k+\alpha_k}} \int_{-\infty}^{\infty} \int_0^\infty \frac{u^{d_k} |\theta|^{-\alpha_k} du d\theta}{[(\theta - 1)^2 + u^2][(\theta + 1)^2 + u^2]^{n_k/2}}.$$

3. If $n_k - 2 < d_k < n_k - 2 + \alpha_k$ and $\theta_k^0 \notin \{0, \pi\}$, then near $\lambda = \pm \theta_k^0$

$$f_k(\lambda) \sim \frac{\varphi_k(1) \psi_k(\theta_k^0) [2 \sin(\theta_k^0)]^{-n_k}}{|\lambda \mp \theta_k^0|^{n_k-2-d_k+\alpha_k}} \int_{-\infty}^{\infty} \frac{|\theta|^{-\alpha_k} d\theta}{|\theta \mp 1|^{n_k-1-d_k}} \int_0^\infty \frac{u^{d_k} du}{(1+u^2)^{n_k/2}}.$$

We do not give the proof of these lemmas, since they are similar to those given for Lemma 3 and Lemma 4 for the continuous time case, we referred to Section 3.

When the measures dR_k are concentrated near of the boundary

$$\delta D^p = \left\{ y : \sup_{1 \leq k \leq p} |y_k| = 1 \right\}$$

of D^p , then the first n terms in (9), or in (10), can only produce a singularity in F , or respectively in H , at the frequencies 0 or π , while singularities at other frequencies can be provided by the last terms. So, for the study of the long memory, the behavior of the measures dR_k near the boundary δD is essential.

If the mixture probabilities $dQ_k(\theta)$ are regular or very diffuse, for instance the Lebesgue measure on some finite interval, and if their supports do not intersect $\{0, \pi\}$, then the long memory induced by the ρ concentration near 1 can "disappear"; i.e., it is not enough to have the mixture probabilities dR_k concentrated near δD^p to reach LM by aggregation of the random parameters $AR(p)$ processes. In fact when the aggregation process exists and dR_k are concentrated near δD^p then is sufficient that the probabilities dQ_k are close to probabilities with support of Lebesgue measure 0. This is a new result.

In the following we characterize the probability measures dR and dQ in order to make condition (3) hold and to obtain the long memory property of aggregation process X . First, we study in detail the case $p = 1$ and $p = 2$, and then we give an example in the case $p = 2$ where the long memory "disappears" by randomness of the parameter θ . Finally, we present the general result in the case of $AR(p)$ processes.

2.1 Case of $AR(1)$ processes

In this section we study the aggregation of random parameter $AR(1)$ processes considering dependence between individual innovations in order to show the influence of interactive innovations on the construction of LM processes.

From convergence results given in [4], we have that a necessary and sufficient condition to obtain the existence of aggregation process is that the interaction correlation χ has a limit in the Cesaro sense. In this case the limit s is such that $-\frac{1}{2} \leq s \leq \infty$. Thus we consider the following two types of interactions.

- Weak interaction: when $-\frac{1}{2} \leq s < \infty$. For instance, short interaction such that $\sum_j |\chi(j)| < \infty$, or large range moderate oscillation when $\sum_j \chi(j) < \infty$ and $\sum_j |\chi(j)| = \infty$.
- Strong interaction: when $s = \infty$. For instance, when $\sum_j \chi(j) < \infty$.

We have that the spectral density F and the transfer function H are given by

$$\begin{aligned} F(\lambda) &= \int_0^1 \frac{\sigma^2}{|1 - \rho e^{i(\lambda - \theta_0)}|^2} dR(\rho) . \\ H(\lambda) &= \int_0^1 \frac{\sigma}{(1 - \rho e^{i(\lambda - \theta_0)})} dR(\rho) . \end{aligned}$$

When dR is concentrated near enough the boundary $\delta D = \{1\}$ of D , we can produce a singularity on F and on H at the frequencies $\theta_0 \in \{0, \pi\}$. By taking, $dR(\rho)$ as in (7) and applying Lemma 1 we can verify that

- If $-1 < d < 1$, then near $\lambda = \theta_0$ $F(\lambda) \cong \frac{1}{|\lambda - \theta_0|^{1-d}}$.
- If $-1 < d < 0$, then near $\lambda = \theta_0$ $|H(\lambda)|^2 \cong \frac{1}{|\lambda - \theta_0|^{-2d}}$.

Furthermore, we have that $\int F(\lambda) d\lambda < \infty$ if and only if $d > 0$ and $\int |H|^2(\lambda) d\lambda < \infty$ if and only if $d > -\frac{1}{2}$. Then we obtain the following theorem.

Theorem 1 [Aggregation of $AR(1)$ processes and long memory.] *If we consider the aggregation of $AR(1)$ processes with random parameter y satisfying Assumption B1, then we have*

1. **Independent innovation case:** *the aggregation X exists if and only if $d > 0$ and X is a long memory process if and only if $d < 1$.*
2. **Common innovation case:** *the aggregation X exists if and only if $d > -\frac{1}{2}$ and it is a long memory process if and only if $d < 0$.*
3. **Interactive innovation case:**
 - 3.1. *Weak interaction: the aggregation X exists if and only if $d > 0$ and it is a long memory process if and only if $d < 1$. In this case $|H|^2$ does not produce long memory.*
 - 3.2. *Strong interaction: we obtain the same result that for common innovation.*

From the above result follow two qualitative ways of obtaining α -LM processes, for $0 < \alpha < 1$; i.e. LM processes with spectral density $G(\lambda)$ such that $G(\lambda) \sim \frac{1}{|\lambda - \theta_0|^\alpha}$ near $\lambda = \theta_0$:

1. If $0 < d < 1$ then considering weak interaction between innovations, we can obtain by aggregation α -LM processes with $0 < \alpha < 1$ from F contribution. In this case $|H|^2$ does not produce LM.
2. If $-\frac{1}{2} < d < 0$ then considering strong long interaction between innovations, we can also obtain by aggregation α -LM processes with $0 < \alpha < 1$, from $|H|^2$ contribution but for a much stronger concentration of the mixture measure near δD .

2.2 Case of $AR(2)$ processes

In this section we give a complete analysis of aggregation of $AR(2)$ under Assumption B1. We consider two type of $AR(2)$ processes. The first type are $AR(2)$ processes with different real poles $y_1 = \rho_1$, $y_2 = \rho_2$. The second type are $AR(2)$ processes with complex conjugated random poles $y_1 = \rho e^{i\theta}$, $y_2 = \rho e^{-i\theta}$ (for which we obtain the particular case of doubly real poles when $\theta \in \{0, \pi\}$).

Case 1: Different real poles

In this case we consider $\rho_1 \neq \rho_2$, $\theta_i^0 \in \{0, \pi\}$ and $\beta_i = 1$ for $i = 1, 2$. Then

$$\begin{aligned} F(\lambda) &= \int_0^1 \frac{\sigma^2}{|1 - \rho e^{i(\lambda - \theta_1^0)}|^2} dR_1(\rho) \int_0^1 \frac{\sigma^2}{|1 - \rho e^{i(\lambda - \theta_2^0)}|^2} dR_2(\rho) . \\ H(\lambda) &= \int_0^1 \frac{\sigma}{(1 - \rho e^{i(\lambda - \theta_1^0)})} dR_1(\rho) \int_0^1 \frac{\sigma}{(1 - \rho e^{i(\lambda - \theta_2^0)})} dR_2(\rho) . \end{aligned}$$

Taking, $dR_i(\rho)$ as in (7) and applying Lemma 1 we can verify that

- i. If $-1 < d_1, d_2 < 1$, and $\theta_1^0 \neq \theta_2^0$, then near $\lambda = \theta_i^0$, for $i = 1, 2$, $F(\lambda) \cong \frac{1}{|\lambda - \theta_i^0|^{1-d_i}}$.
- i'. If $-1 < d_1, d_2 < 1$, and $\theta_1^0 = \theta_2^0$, then near $\lambda = \theta_1^0$, $F(\lambda) \cong \frac{1}{|\lambda - \theta_1^0|^{2-d_1-d_2}}$.
- ii. If $-1 < d_1, d_2 < 0$, and $\theta_1^0 \neq \theta_2^0$, then near $\lambda = \theta_i^0$, for $i = 1, 2$, $|H(\lambda)| \cong \frac{1}{|\lambda - \theta_i^0|^{-d_i}}$.
- ii'. If $-1 < d_1, d_2 < 0$, and $\theta_1^0 = \theta_2^0$, then near $\lambda = \theta_i^0$, for $i = 1, 2$, $|H(\lambda)| \cong \frac{1}{|\lambda - \theta_1^0|^{-d_1-d_2}}$.

On the other hand, when $\theta_1^0 \neq \theta_2^0$ we have that $\int F(\lambda) d\lambda < \infty$ if and only if $d_1 > 0, d_2 > 0$ and $\int |H|^2(\lambda) d\lambda < \infty$ if and only if $d_1 > -\frac{1}{2}, d_2 > -\frac{1}{2}$.

When $\theta_1^0 = \theta_2^0$, $\int F(\lambda) d\lambda < \infty$ if and only if $d_1 + d_2 > 1$ and $\int |H|^2(\lambda) d\lambda < \infty$ if and only if $d_1 + d_2 > -1$.

Case 2: Complex conjugated poles

In this case we consider two complex conjugated poles $\rho e^{i\theta}$, $\rho e^{-i\theta}$.

$$\begin{aligned} F(\lambda) &= \int_0^1 \int_{-\pi}^{\pi} \frac{\sigma^2}{|1 - \rho e^{i(\lambda-\theta)}|^2 |1 - \rho e^{i(\lambda+\theta)}|^2} dR(\rho) dQ(\theta) . \\ H(\lambda) &= \int_0^1 \int_{-\pi}^{\pi} \frac{\sigma^2}{(1 - \rho e^{i(\lambda-\theta)})(1 - \rho e^{i(\lambda+\theta)})} dR(\rho) dQ(\theta) . \end{aligned}$$

We take $dR(\rho)$ as in (7) and Q as in (8). Then, applying Lemma 1 and Lemma 2 we obtain

- i. If $-1 < d < 2 + \beta$, $\beta \leq 1$ and $\theta^0 \in \{0, \pi\}$, then near $\lambda = \theta^0$ $F(\lambda) \cong \frac{1}{|\lambda - \theta^0|^{2+\beta-d}}$.
- i'. If $-1 < d < \beta$, $\beta \leq 1$ and $\theta^0 \notin \{0, \pi\}$, then near $\lambda = \theta^0$ $F(\lambda) \cong \frac{1}{|\lambda - \theta^0|^{\beta-d}}$.
- ii. If $-1 < d < \beta$, $\beta \leq 1$ and $\theta^0 \in \{0, \pi\}$, then near $\lambda = \theta^0$ $|H(\lambda)| \cong \frac{1}{|\lambda - \theta^0|^{\beta-d}}$.
- ii'. If $-1 < d < \beta - 1$, $\beta \leq 1$ and $\theta^0 \notin \{0, \pi\}$, then near $\lambda = \pm\theta^0$ $|H(\lambda)| \cong \frac{1}{|\lambda - \theta^0|^{-1+\beta-d}}$.

On the other hand, when $\theta^0 \in \{0, \pi\}$ we have that $\int F(\lambda) d\lambda < \infty$ if and only if $d > 1 + \beta$ and $\int |H|^2(\lambda) d\lambda < \infty$ if and only if $d > \beta - \frac{1}{2}$.

When $\theta^0 \notin \{0, \pi\}$, then $\int F(\lambda) d\lambda < \infty$ if and only if $d > -1 + \beta$ and $\int |H|^2(\lambda) d\lambda < \infty$ if and only if $d > \beta - \frac{3}{2}$.

Finally, we can summarize these results in the following theorem.

Theorem 2 [Aggregation of AR(2) processes and long memory.] *If we consider the aggregation of AR(2) processes with random parameters satisfying Assumption B1, then we have*

1. Independent innovations case:

Different real poles: $\beta_i = 1$, $\theta_i \in \{0, \pi\}$ for $i = 1, 2$.

- $\theta_1^0 = \theta_2^0$, X exists if and only if $d_1 + d_2 > 1$, and it is a LM process if and only if $d_1, d_2 < 1$.
- $\theta_1^0 \neq \theta_2^0$, X exists if and only if $d_1, d_2 > 0$, and it is a LM process if and only if $\min\{d_1, d_2\} < 1$.

Complex conjugated poles: $\beta \leq 1$.

- $\theta^0 \in \{0, \pi\}$, X exists if and only if $d > 1 + \beta$, and it is a LM process if and only if $d < 2 + \beta$.
- $\theta^0 \notin \{0, \pi\}$, X exists if and only if $d > -1 + \beta$, and it is a LM process if and only if $d < \beta$.

2. Common innovation case:

Different real poles: $\beta_i = 1$, $\theta_i \in \{0, \pi\}$ for $i = 1, 2$.

- $\theta_1^0 = \theta_2^0$, X exists if and only if $d_1 + d_2 > -1$, and it is a LM process if and only if $d_1, d_2 < 0$.
- $\theta_1^0 \neq \theta_2^0$, X exists if and only if $d_1, d_2 > -\frac{1}{2}$, and it is a LM process if and only if $\min\{d_1, d_2\} < 0$.

Complex conjugated poles: $\beta \leq 1$.

- $\theta^0 \in \{0, \pi\}$, X exists if and only if $d > \beta - \frac{1}{2}$, and it is a LM process if and only if $d < \beta$.
- $\theta^0 \notin \{0, \pi\}$, X exists if and only if $d > \beta - \frac{3}{2}$, and it is a LM process if and only if $d < \beta - 1$.

Remark 3 *For different real random poles with distribution concentrate enough near $\rho = 1$ we find the same long memory behavior in $\mathbb{M}(\text{AR}(2))$ as for $\mathbb{M}(\text{AR}(1))$.*

2.2.1 Opposite phenomena: disappearance of long memory by randomness of θ parameter

Now we illustrate how the long memory can "disappear" by randomness of θ . Here we only consider the case of complex and not real random poles $\rho e^{i\theta}$, $\rho e^{-i\theta}$ and independent innovations. This result can be easily generalized to the case of interactive innovations. We denote by \bar{S}_Q the closed support of Q .

As we have already mentioned when $\beta = 1$ we consider that the measure Q is a Dirac's delta; i.e. $Q_k(\theta) = \delta(\theta - \theta^0)$. For $0 < \beta < 1$ the measure $dQ(\theta) = \psi(\theta)|\theta - \theta^0|^{-\beta}d\theta$ is strongly concentrated near θ^0 and for $\beta \leq 0$ this measure is regular. In the particular case of $\beta = 0$ we consider that ψ is such that Q is a diffuse measure. For instance, if we take $\psi(\theta) = \mathbb{1}_{(\tau_1, \tau_2)}(\theta)$ then Q is a uniform measure on (τ_1, τ_2) . With respect to dR , we have that when $-1 < d < 0$ this measure is strongly concentrated near $\rho = 1$, and when $d \geq 0$ then dR is a regular measure.

In Theorem 2 we have given conditions under parameters β and d in order to obtain the existence and the long memory property of the aggregation process by means of the aggregation of $AR(2)$ processes. We resume this result in the case of random poles and independent innovations in Figure 1 and Figure 2, where we show the values of α -LM parameter obtained.

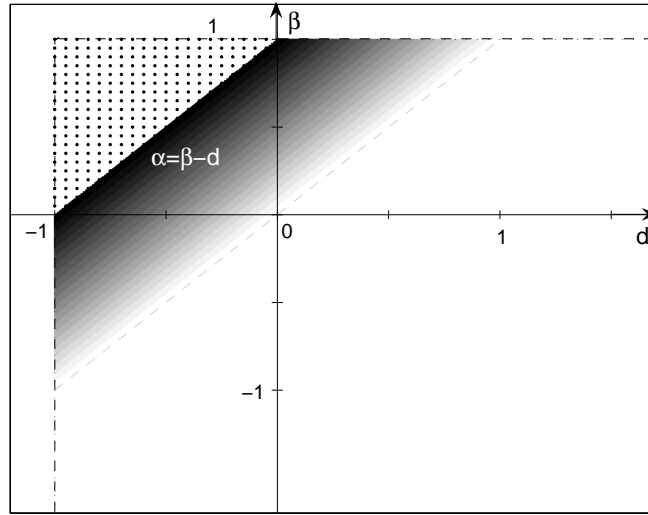


Figure 1: Values of α -LM parameter (case: \bar{S}_Q does not intersect $\{0, \pi\}$).

Figure 1 represents the existence of the aggregation and the LM property in terms of the values of (d, β) in the case where \bar{S}_Q does not intersect $\{0, \pi\}$. We recall that we always consider $d \geq -1$ and $\beta \leq 1$. The dotted region corresponds to the values of the parameter (d, β) for which the aggregation does not exist. The white region corresponds to values where the aggregation exists but there is no long memory. The remaining region corresponds to the values for which we obtain the existence and LM property. In this last region we plot the α -LM parameter, which is given by $\alpha = \beta - d$. Here, black stands for the maximum value of α (in our case $\alpha = 1$, corresponding to an aggregation process strongly dependent) and white stands for its minimum value ($\alpha = 0$, coinciding with the case where the aggregation exists but there is no long memory).

We can appreciate that for a value of d fixed, if β is small enough, then the LM "disappears", i.e. we can take a measure Q regular enough that we do not get LM property. For instance, for $0 < d < 1$ fixed, if we take Q as a Lebesgue measure (i.e. $\beta = 0$) such that \bar{S}_Q does not intersect $\{0, \pi\}$, then we do not obtain the long memory.

Roughly speaking, it is not enough to have the mixture probability concentrated near $\delta D =$

$\{\rho = 1\}$ to reach LM by aggregation. But if we take a measure Q such that it is close to a probability with support of Lebesgue measure 0, for instance $\beta > 0$, then we obtain the LM . On the other hand, if the measure dR is very concentrated near $\rho = 1$, i.e. $d \rightarrow -1$, then it is necessary to take a regular measure Q ($\beta < 0$) so that we get the existence of aggregation.

When \bar{S}_Q intersects the set $\{0, \pi\}$, we have that $\alpha = \beta - (d - 2)$, and we obtain a similar behaviour to the precedent case. We note that in this case the corresponding graphic is the same but translated two units in d -axis, see Figure 2.

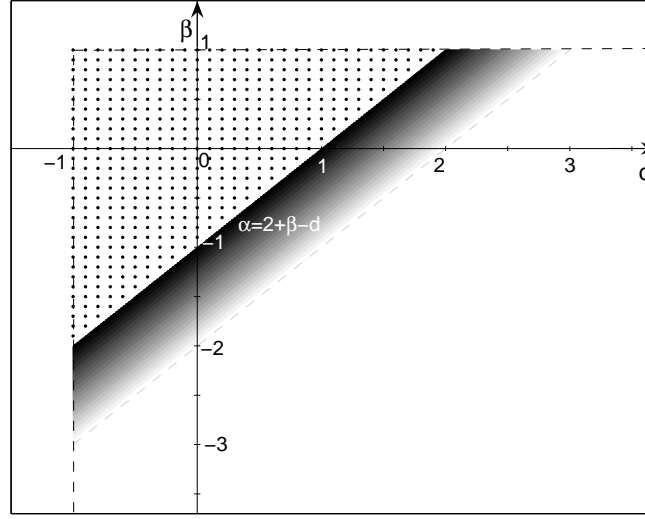


Figure 2: Values of α -LM parameter (case: 0 or π in \bar{S}_Q).

2.3 General case of $AR(p)$ processes

Now, we present the general result in the case of $AR(p)$ processes that allows us to give the condition under the measures dR and dQ in order to obtain the existence and the long memory of the aggregation process. This result is based on Lemma 1 and Lemma 2.

We denote $n_k = 2m_k$ in the case of independent innovations and $n_k = m_k$ in the common innovation case.

Theorem 3 [Aggregation of $AR(p)$ processes and long memory.] *Let $\{Z^i : i \in \mathbb{N}\}$ be a sequence of $AR(p)$ processes defined as in (5), let $\{\rho^j\}$ and $\{\theta^j\}$ be the corresponding sequences of random parameter vectors, with $\rho^j = (\rho_1^j, \dots, \rho_q^j)$ and $\theta^j = (\theta_1^j, \dots, \theta_q^j)$ for $j \in \mathbb{N}$. Let ρ and θ satisfying Assumption B1. Then, the aggregation exists if and only if the three following conditions hold:*

- $\sum_{k \in K_1} n_k (1 + \mathbb{1}_{\{\beta < 1\}}(\beta_k)) - 2 + \beta_k - d_k < 1$.
- $\sum_{k \in K_2} n_k (1 + \mathbb{1}_{\{\beta < 1\}}(\beta_k)) - 2 + \beta_k - d_k < 1$.
- $\min_{k \in K_3} \{n_k - 2 + \beta_k - d_k\} < 1$

where $K_1 = \{k : \theta_k^0 = 0\}$, $K_2 = \{k : \theta_k^0 = \pi\}$ and $K_3 = \{k : \theta_k^0 \notin \{0, \pi\}\}$.

Moreover, X is a long memory process if and only if some of following conditions is satisfied:

- $\forall k \in K_1, n_k \mathbb{1}_{\{\beta < 1\}}(\beta_k) - 1 < d_k < n_k (1 + \mathbb{1}_{\{\beta < 1\}}(\beta_k)) - 2 + \beta_k$.
- $\forall k \in K_2, n_k \mathbb{1}_{\{\beta < 1\}}(\beta_k) - 1 < d_k < n_k (1 + \mathbb{1}_{\{\beta < 1\}}(\beta_k)) - 2 + \beta_k$.
- There exists $k \in K_3$ such that $n_k - 2 < d_k < n_k - 2 + \beta_k$.

The proof of this theorem is similar to the proof that we will give for the continuous case.

3 Aggregation of $OU(p)$ processes and long memory

The purpose of this sections is to establish the analogues results to Section 2 for the continuous time case.

We consider elementary processes Z^i as stationary solutions of p -order linear stochastic differential equations $LSDE(p)$ driven by a standard Brownian motion W^j . We call these elementary processes Ornstein-Uhlenbeck processes of order p , $OU(p)$.

We consider that the characteristic polynomial associated to $LSDE(p)$ equation is factorized as

$$A(s, y) = \prod_{k=1}^p (s + y_k)$$

where, $y = (y_1, \dots, y_p)$ is the vector of random roots, whose distribution has support $(0, \infty)^p$. Let $c_s(y)$ be the inverse Laplace transform of $A(s, y)^{-1}$ defined on \mathbb{R}^+ . We define the $OU(p)$ process with characteristic polynomial $A(s, y)$ by

$$Z_t(y) := \int_{-\infty}^t c_s(y) dW_s, \quad t \in \mathbb{R}^+. \quad (11)$$

This process is, $\mu - a.s.$, a stationary centered Gaussian process with spectral density $g(\lambda, y) = \sigma^2 |A(s, y)|^{-2}$. We consider the transfer function $h(\lambda, y) = \sigma A(s, y)^{-1}$.

If F is the mixture given by $F(\lambda) = \mathbb{E}[g(\lambda, y)]$, then the process given in (11) is well defined, when y is a random vector, if the distribution μ of y satisfies condition (3).

Let us consider a sequence $\{Z_t^i : i \in \mathbb{N}\}$ of $OU(p)$ processes and note by $\{W^i\}$ and $Y = \{y^i\}$ the corresponding sequences of Gaussian innovations and random parameter vectors. We assume that $\{W^i\}$ and Y satisfy the Assumption A1 and that interaction χ satisfies the hypothesis of Theorem 1 given in [4].

We denote by $\mathbb{OU}(p)$ the class of $OU(p)$ processes and by $\mathbb{M}(\mathbb{OU}(p))$ the class of processes that can be obtained by aggregation of elementary processes in $\mathbb{OU}(p)$.

In the sequel, we consider that

$$A(s, y) = \prod_{k=1}^n (s + r_k)^{m_k} \prod_{k=n+1}^q [(s + r_k + i\tau_k)(s + r_k - i\tau_k)]^{m_k},$$

is a polynomial of grade p with n real roots and $2(q - n)$ complex pairwise conjugate roots having strictly positive real parts, where $n \leq q$, m_k is the multiplicity of the roots $y_k = r_k \pm i\tau_k$, with $\tau_k = 0$ for $1 \leq k \leq n$, and $p = \sum_{k=1}^n m_k + 2 \sum_{j=n+1}^q m_j$.

For $r = (r_1, \dots, r_q)$ and $\tau = (\tau_1, \dots, \tau_q)$ fixed, under condition (3) Z_t is a stationary centered Gaussian process with spectral density

$$g(\lambda, r, \tau) = \prod_{k=1}^n \frac{1}{(\lambda^2 + r_k^2)^{m_k}} \prod_{k=n+1}^q \frac{1}{[(\lambda - \tau_k)^2 + r_k^2][(\lambda + \tau_k)^2 + r_k^2]^{m_k}}, \quad (12)$$

and transfer function

$$h(\lambda, r, \tau) = \prod_{k=1}^n \frac{1}{(i\lambda + r_k)^{m_k}} \prod_{k=n+1}^q \frac{1}{[(\lambda - \tau_k)i + r_k][(\lambda + \tau_k)i + r_k]^{m_k}}. \quad (13)$$

As in the discrete case, we will consider the following assumption.

Assumption B2: Let r_1, \dots, r_q be independent random variables, such that r_k has distribution

$$dR_k(r) = |r|^{d_k} \varphi_k(r) dr, \quad (14)$$

where φ_k is a bounded positive function, continuous in $r = 0$ with $\varphi_k(0) > 0$. Let τ_1, \dots, τ_q be independent random variables and independent of r_1, \dots, r_q , such that τ_k has distribution

$$dQ_k(\tau) = \frac{\psi_k(\tau)d\tau}{|\tau - \tau_k^0|^{\beta_k}}, \quad (15)$$

where $\beta_k \leq 1$ ψ_k is a bounded positive function, continuous in $\tau = \tau_k^0$ and $\psi_k(\tau_k^0) > 0$. By convention when $\beta_k = 1$ we will consider $Q_k(\tau) = \delta(\tau - \tau_k^0)$ and $\tau_k^0 = 0$ for $1 \leq k \leq n$.

Let us now study the local behavior of the spectral densities F and $|H|^2$, defined by equations (2) and (4) respectively. We take $n_k = 2m_k$ in the case of independent innovations and $n_k = m_k$ in the common innovation case. Then, under Assumption B2 we obtain the following technical lemmas.

Lemma 3

1. If $-1 < d_k < n_k - 1$, then near $\lambda = 0$

$$\int_0^\infty \frac{|r|^{d_k} \varphi_k(r)}{(\lambda^2 + r^2)^{n_k/2}} dr \sim \frac{\varphi_k(0)}{|\lambda|^{n_k-1-d_k}} \int_0^\infty \frac{u^{d_k} du}{(1+u^2)^{n_k/2}}.$$

2. If $-1 < d_k < n_k - 1$ and $\tau_k^0 \neq 0$, then near $\lambda = \pm \tau_k^0$

$$\int_0^\infty \frac{|r|^{d_k} \varphi_k(r) dr}{[(\lambda - \tau_k^0)^2 + r^2][(\lambda + \tau_k^0)^2 + r^2]^{n_k/2}} \sim \frac{\varphi_k(0)(2\tau_k^0)^{-n_k}}{|\lambda \mp \tau_k^0|^{n_k-1-d_k}} \int_0^\infty \frac{u^{d_k} du}{(1+u^2)^{n_k/2}}.$$

Proof The first part is shown by making the variable change $r = |\lambda|u$ and then taking limits as $\lambda \rightarrow 0$. The condition $-1 < d_k < n_k - 1$ implies the convergence of the following integral $\int_0^\infty \frac{u^{d_k}}{(1+u^2)^{n_k/2}} du$.

The second part can be proved in the same way, with the variable change $r = |\lambda \mp \tau_k^0|u$ and taking limits as $\lambda \rightarrow \pm \tau_k^0$. □

Lemma 4 Let $\alpha_k < 1$ and

$$f_k(\lambda) = \int_{-\infty}^\infty \int_0^\infty \frac{|r|^{d_k} \varphi_k(r) \psi_k(\tau) |\tau - \tau_k^0|^{-\alpha_k} dr d\tau}{[(\lambda - \tau)^2 + r^2][(\lambda + \tau)^2 + r^2]^{n_k/2}}.$$

1. If $n_k - 1 < d_k < 2n_k - 2 + \alpha_k$ and $\tau_k^0 = 0$, then near $\lambda = 0$

$$f_k(\lambda) \sim \frac{\varphi_k(0)\psi_k(0)}{|\lambda|^{2n_k-2-d_k+\alpha_k}} \int_{-\infty}^\infty \int_0^\infty \frac{u^{d_k} |\theta|^{-\alpha_k} du d\theta}{[(\theta - 1)^2 + u^2][(\theta + 1)^2 + u^2]^{n_k/2}}.$$

2. If $n_k - 2 < d_k < n_k - 2 + \alpha_k$ and $\tau_k^0 \neq 0$, then near $\lambda = \pm \tau_k^0$

$$f_k(\lambda) \sim \frac{\varphi_k(0)\psi_k(\tau_k^0)(2\tau_k^0)^{-n_k}}{|\lambda \mp \tau_k^0|^{n_k-2+\alpha_k-d_k}} \int_{-\infty}^\infty \frac{|\theta|^{-\alpha_k} d\theta}{|\theta \mp 1|^{n_k-1-d_k}} \int_0^\infty \frac{u^{d_k} du}{(1+u^2)^{n_k/2}}.$$

Proof We show point 1 by making the variable changes $r = |\lambda|u$ and $\tau = \lambda\theta$ and then taking limits as $\lambda \rightarrow 0$. The result holds if the integral

$$I := \int_{-\infty}^\infty \int_0^\infty \frac{u^{d_k} du d\theta}{[(\theta - 1)^2 + u^2][(\theta + 1)^2 + u^2]^{n_k/2} |\theta|^{\alpha_k}}$$

is convergent. To verify that we will use that

$$u^2(1 + \theta^2 + u^2) < ((\theta - 1)^2 + u^2)((\theta + 1)^2 + u^2)$$

and then we will make the following variable change: $u = (1 + \theta^2)^{\frac{1}{2}}r$, from where

$$I \leq \int_{-\infty}^{\infty} \int_0^{\infty} \frac{u^{d_k - n_k} du d\theta}{(1 + \theta^2 + u^2)^{n_k/2} |\theta|^{\alpha_k}} = \int_0^{\infty} \frac{2\theta^{-\alpha_k} d\theta}{(1 + \theta^2)^{\frac{2n_k - 1 - d_k}{2}}} \int_0^{\infty} \frac{r^{d_k - n_k} dr}{(1 + r^2)^{n_k/2}}.$$

Finally we have that these two last integrals converge if $\alpha_k < 1$ and $n_k - 1 < d_k < 2n_k - 2 + \alpha_k$.

To prove point 2 we make two variable changes $r = |\lambda \mp \tau|u$ and $\tau - \tau_k^0 = (\lambda \mp \tau_k^0)\theta$, which give us

$$\begin{aligned} f_k(\lambda) &= \frac{1}{|\lambda \mp \tau_k^0|^{(n_k - 2) + \alpha_k - d_k}} \int_{-\infty}^{\infty} \frac{\psi_k(\tau_k^0 + (\lambda \mp \tau_k^0)\theta)}{|\theta \mp 1|^{n_k - 1 - d_k} |\theta|^{\alpha_k}} \\ &\quad \times \int_0^{\infty} \frac{u^{d_k} \varphi_k(|\lambda \mp \tau_k^0| |\theta \mp 1| u) du d\theta}{[(1 + u^2)((\lambda \pm \tau_k^0) \pm (\lambda \mp \tau_k^0)\theta)^2 + ((\lambda \mp \tau_k^0)(\theta \mp 1)u)^2]^{n_k/2}}. \end{aligned}$$

Then, taking limits as $\lambda \rightarrow \pm \tau_k^0$,

$$f_k(\lambda) \sim \frac{\varphi_k(0) \psi_k(\tau_k^0) (2\tau_k^0)^{-n_k}}{|\lambda \mp \tau_k^0|^{n_k - 2 + \alpha_k - d_k}} \int_{-\infty}^{\infty} \frac{d\theta}{|\theta \mp 1|^{n_k - 1 - d_k} |\theta|^{\alpha_k}} \int_0^{\infty} \frac{u^{d_k} du}{(1 + u^2)^{n_k/2}}.$$

Finally, we can see that these two integrals converge if $\alpha_k < 1$ and $n_k - 2 < d_k < n_k - 2 + \alpha_k$. \square

From Lemma 3 and Lemma 4 and under Assumption B2 we show the following theorem which gives the condition over parameters d, β that allows us to obtain the long memory for the aggregation of $OU(p)$ processes.

Theorem 4 [Aggregation of $OU(p)$ processes and long memory.] *Let $\{Z^i : i \in \mathbb{N}\}$ be an i.i.d sequence of $OU(p)$ processes defined as in (11), let $\{W^i\}$, $\{r^i\}$ and $\{\tau^i\}$ be the corresponding i.i.d sequences of Brownian motions and of random parameters vectors. Let r and τ be random vector satisfying Assumption B2. Then, the aggregation exists if condition (3) holds and there is long memory if and only if*

$$d_k < n_k(1 + \mathbb{1}_{\{0\}}(\tau_k^0) \mathbb{1}_{\{\beta < 1\}}(\beta_k)) - 2 + \beta_k, \quad \text{for some } k \in \{1, \dots, q\}.$$

Proof It is clear that condition (3) implies the aggregation existence. On the other hand, as long memory of the aggregation is related to the spectral density singularities, then by Lemma 3 and Lemma 4, the result holds. \square

This theorem generalizes the results given by other authors in the following sense: in our approach τ is a random parameters vector and not a constant vector, moreover we consider multiple roots. In the case where $\beta_k = 1$ for $1 \leq k \leq q$, i.e. when we consider the parameters vector τ fixed, the necessary and sufficient conditions for the aggregation existence can be easily written in terms of the parameters d_k and n_k , as we can see in the Corollary 1. In the general case, the aggregation existence does not depend on the decay of functions φ_k and ψ_k , for $1 \leq k \leq q$. For instance, it can be seen that in the case of $OU(2)$ processes if we consider φ_k bounded and $\psi_k(\tau) \sim |\tau|^{-\beta_k} \wedge |\tau|^{-\beta'_k}$ with $\beta_k < 1$ and $\beta'_k > 0$, then the aggregation exists if and only if $1 + \beta_k < d_k < 2$.

The following corollary generalize the result given in [12], because these authors consider only the case of simple roots and τ fixed.

Corollary 1 *Let $\{Z^i : i \in \mathbb{N}\}$ be an i.i.d sequence of $OU(p)$ processes, $\{W^i\}$, $\{r^i\}$ and $\{\tau^i\}$ the corresponding i.i.d sequences of Brownian motions and of parameters vectors. If we consider $\alpha_k = 1$ for $1 \leq k \leq q$ and the parameters vectors r and τ satisfying Assumption B2, then the aggregation exists and there is long memory if and only if the following conditions hold:*

- $-1 < d_k < n_k - 1$ for $1 \leq k \leq n$.
- $n_k - 2 < d_k < n_k - 1$, for $n < k \leq q$.
- $\sum_{k=1}^n n_k - d_k < n + 1$.

Proof By applying Lemma 3 and if $-1 < d_k < n_k - 1$, for $1 \leq k \leq q$, then near to $\lambda = 0$ we have that the spectral density G of the aggregation process is such that

$$G(\lambda) \sim \prod_{k=1}^n \frac{1}{|\lambda|^{n_k-1-d_k}} \varphi_k(0) \int_0^\infty \frac{u^{d_k}}{(1+u^2)^{\frac{n_k}{2}}} du,$$

and near to $\lambda = \pm \tau_k^0$

$$G(\lambda) \sim \frac{1}{|\lambda \mp \tau_k^0|^{n_k-1-d_k}} \frac{\varphi_k(0)}{(2\tau_k^0)^{n_k}} \int_0^\infty \frac{u^{d_k} du}{(1+u^2)^{\frac{n_k}{2}}}.$$

Moreover, we have that

$$G(\lambda) \leq \frac{1}{\prod_{k=1}^n \lambda^{n_k} \prod_{k=n+1}^q (\lambda^2 - \tau_k^{0^2})^{n_k}},$$

which allows us to bound G when $\lambda \rightarrow \pm\infty$. Then, the aggregation exists if and only if $-1 < d_k < n_k - 1$ for $1 \leq k \leq q$, $n_k - 1 - d_k < 1$ for $n < k \leq q$, and $\sum_{k=1}^n n_k - 1 - d_k < 1$. Furthermore, there is long memory if and only if $n_k - 1 - d_k > 0$ for some $k \in \{1, \dots, q\}$, from where the theorem holds. \square

A slight modification in the proof of Theorem 4 allows us to extend these results when we consider the measures dQ_k as

$$dQ_k(\tau) = \sum_{j=1}^{n_k} p_j \delta(\tau - \tau_k^j) + \Psi_k(\tau) d\tau$$

where Ψ_k is a positive function with singular points s_1, \dots, s_{l_k} such that for each s_j there exists a function $\psi_{j,k}$, regular in a neighborhood $V(s_j)$ of s_j , such that

$$\Psi_k(s) \sim \frac{\psi_{j,k}(s)}{|s - s_j|^{\beta_{j,k}}}, \quad \text{for } s \in V(s_j)$$

and Ψ_k a function bounded out of $\bigcup_j V(s_j)$.

Remark 4 *In the aggregation of OU processes the phenomenon of disappearance of long memory also can happen by randomness of parameter τ . The analysis is similar to the one given for the discrete time case.*

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